DUAL OF BASS NUMBERS AND DUALIZING MODULES

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Abstract. Let R be a Noetherian ring and let C be a semidualizing R-module. In this paper, we impose various conditions on C to be dualizing. For example, as a generalization of Xu [22, Theorem 3.2], we show that C is dualizing if and only if for an R-module M, the necessary and sufficient condition for M to be C-injective is that $\pi_i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $i \neq \operatorname{ht}(\mathfrak{p})$, where π_i is the invariant dual to the Bass numbers defined by E.Enochs and J.Xu [8].

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity. A finitely generated R-module C is semidualizing if the natural homothety map $R \longrightarrow \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}^i_R(C, C) = 0$ for all $i > 0$. Semidualizing modules have been studied by Foxby [9], Vasconcelos [20] and Golod [10] who used the name *suitable* for these modules. Dualizing complexes, introduced by A.Grothendieck, is a powerful tool for investigating cohomology theories in algebraic geometry. A bounded complex of R-modules D with finitely generated homologies is said to be a dualizing complex for R, if the natural homothety morphism $R \to \mathbf{R}\operatorname{Hom}_R(D, D)$ is quasiisomorphism, and $\operatorname{id}_R(D) < \infty$. These notion has been extended to semidualizing complexes by L.W. Christensen [5]. A bounded complex of R-modules C with finitely generated homologies is semidualizing for R if the natural homothety morphism $R \to \mathbf{R}\operatorname{Hom}_R(C, C)$ is quasiisomorphism. He used these notion to define a new homological dimension for complexes, namely G_C-dimension, which is a generalization of Yassemi's G-dimension [23]. The following, is the translation of a part of [5, Proposition 8.4] to the language of modules:

Theorem 1. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let C be a semidualizing R-module. The following are equivalent:

(i) C is dualizing.
(ii) $\mathrm{G}_C\text{-}\dim_R(M) < \infty$ for all finite R-modules M.
(iii) $\mathrm{G}_C\text{-}\dim_R(k) < \infty$.

In particular, the above theorem recovers [4, 1.4.9]. Note that k is a Cohen-macaulay R-module of type 1. R.Takahashi, in [17, Theorem 2.3], replaced the condition $\mathrm{G}\text{-}\dim_R(k) < \infty$ in [4, 1.4.9] by weaker conditions and obtained a nice characterization for Gorenstein rings.

2000 *Mathematics Subject Classification.* 13C05, 13D05, 13D07, 13H10.
Key words and phrases. Semidualizing modules, dualizing modules, G_C-dimension, Bass numbers, dual of Bass numbers, minimal flat resolution, local cohomology.

Indeed, he showed that R is Gorenstein, provided that either R admits an ideal I of finite G-dimension such that R/I is Gorenstein, or there exists a Cohen-Macaulay R-module of type 1 and of finite G-dimension. The following is the main result of section 3, which generalizes Theorem 1 as well as [17, Theorem 2.3]. See Theorem 3.4 below.

Theorem 2. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let C be a semidualizing R-module. The following are equivalent:

(i) C is dualizing.
(ii) There exists an ideal $\mathfrak{a}$ with $\mathrm{G}_C\text{-}\dim_R(\mathfrak{a}C) < \infty$ such that $C/\mathfrak{a}C$ is dualizing for $R/\mathfrak{a}$.
(iii) There exists a Cohen-Macaulay R-module M with $r_R(M) = 1$ and $\mathrm{G}_C\text{-}\dim_R(M) < \infty$.
(iv) $r_R(C) = 1$ and there exists a Cohen-Macaulay R-module M with $\mathrm{G}_C\text{-}\dim_R(M) < \infty$.

E.Enochs et al. [1], solved a long standing conjecture about the existence of flat covers. Indeed, they showed that if R is any ring, then all R-modules have flat covers. E.Enochs [6], determined the structure of flat cotorsion modules. Also, E.Enochs and J.Xu [8, Definition 1.2], defined a new invariant π_i, dual to the Bass numbers, for modules related to flat resolutions. J.Xu [22], studied the minimal injective resolution of flat R-modules and minimal flat resolution of injective R-modules. He characterized Gorenstein rings in terms of vanishing of Buss numbers of flat modules, and vanishing of dual of Bass numbers of injective modules. More precisely, the following theorem is [22, Theorems 2.1 and 3.2].

Theorem 3. Let R be a Noetherian ring. The following are equivalent:

(i) R is Gorenstein.
(ii) An R-module F is flat if and only if $\mu^i(\mathfrak{p}, F) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ whenever $i \neq \mathrm{ht}\,(\mathfrak{p})$.
(iii) An R-module E is injective if and only if $\pi_i(\mathfrak{p}, E) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ whenever $i \neq \mathrm{ht}\,(\mathfrak{p})$.

In section 4, we give a generalization of Theorem 3. Indeed, in Theorem 4.3, we prove the following result.

Theorem 4. Let R be a Noetherian ring and let C be a semidualizing R-module. The following are equivalent:

(i) C is pointwise dualizing.
(ii) An R-module M is C-injective if and only if $\pi_i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ whenever $i \neq \mathrm{ht}\,(\mathfrak{p})$.
(iii) An R-module M is injective if and only if $\pi_i(\mathfrak{p}, \mathrm{Hom}_R(C, M)) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ whenever $i \neq \mathrm{ht}\,(\mathfrak{p})$.

Theorem 4 has several applications. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring possessing a canonical module. In this section, we give the structure of the minimal flat resolution of $\mathrm{H}^d_{\mathfrak{m}}(R)$, the top local cohomology of R. More precisely, the following theorem is Corollary 4.7.

Theorem 5. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring possessing a canonical module. The minimal flat resolution of $\mathrm{H}^d_{\mathfrak{m}}(R)$ is of the form

$$0 \longrightarrow \widehat{R_{\mathfrak{m}}} \longrightarrow \cdots \longrightarrow \prod_{\mathrm{ht}\,(\mathfrak{p})=1} T_{\mathfrak{p}} \longrightarrow \prod_{\mathrm{ht}\,(\mathfrak{p})=0} T_{\mathfrak{p}} \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(R) \longrightarrow 0,$$

in which $T_{\mathfrak{p}}$ is the completion of a free $R_{\mathfrak{p}}$-module with respect to $\mathfrak{p}R_{\mathfrak{p}}$-adic topology.

In this section, by using the above resolution, we obtain the following isomorphism for a d-dimensional Cohen-Macaulay local ring (See Corollary 4.8).

$$\mathrm{Tor}\,_i^R(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{H}^d_{\mathfrak{m}}(R)) \cong \begin{cases} \mathrm{H}^d_{\mathfrak{m}}(R) & i = d, \\ 0 & i \neq d. \end{cases}$$

2. PRELIMINARIES

In this section, we recall some definitions and facts which are needed throughout this paper. By an injective cogenerator, we always mean an injective R-module E for which $\mathrm{Hom}\,_R(M, E) \neq 0$ whenever M is a nonzero R-module. For an R-module M, the injective hull of M, is always denoted by $E(M)$.

Definition 2.1. Let $\mathcal{X}$ be a class of R-modules and M an R-module. An $\mathcal{X}$-*resolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \ldots \longrightarrow X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \ldots \longrightarrow X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for all $n \geq 1$. Also the $\mathcal{X}$-*projective dimension* of M is the quantity

$$\mathcal{X}\text{-pd}\,_R(M) := \inf\{\sup\{n \geq 0 | X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\} \quad .$$

So that in particular $\mathcal{X}$-pd$\,_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension zero are precisely the non-zero modules in $\mathcal{X}$. The terms of $\mathcal{X}$-*coresolution* and $\mathcal{X}$-id are defined dually.

Definition 2.2. A finitely generated R-module C is *semidualizing* if it satisfies the following conditions:

(i) The natural homothety map $R \longrightarrow \mathrm{Hom}\,_R(C, C)$ is an isomorphism.
(ii) $\mathrm{Ext}\,^i_R(C, C) = 0$ for all $i > 0$.

For example a finitely generated projective R-module of rank 1 is semidualizing. If R is Cohen-Macaulay, then an R-module D is dualizing if it is semidualizing and that $\mathrm{id}\,_R(D) < \infty$. For example the canonical module of a Cohen-Macaulay local ring, if exists, is dualizing.

Definition 2.3. Following [12], let C be a semidualizing R-module. We set

$\mathcal{F}_C(R)$ = the subcategory of R–modules $C \otimes_R F$ where F is a flat R–module.
$\mathcal{I}_C(R)$ = the subcategory of R–modules $\mathrm{Hom}\,_R(C, I)$ where I is an injective R–module.

The R-modules in $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ are called *C-flat* and *C-injective*, respectively. If $C = R$, then it recovers the classes of flat and injective modules, respectively. We use the notations C-fd and C-id instead of $\mathcal{F}_C$-pd and $\mathcal{I}_C$-id, respectively.

Proposition 2.4. *Let C be a semidualizing R-module. Then we have the following:*

(i) $\mathrm{Supp}\,(C) = \mathrm{Spec}\,(R)$, $\dim(C) = \dim(R)$ *and* $\mathrm{Ass}\,(C) = \mathrm{Ass}\,(R)$.

(ii) *If $R \to S$ is a flat ring homomorphism, then $C \otimes_R S$ is a semidualizing S-module.*

(iii) *If $x \in R$ is R–regular, then C/xC is a semidualizing R/xR-module.*

(iv) *If, in addition, R is local, then* $\mathrm{depth}_R(C) = \mathrm{depth}\,(R)$.

Proof. The parts (i), (ii) and (iii) follow from the definition of semidualizing modules. For (iv), note that an element of R is R-regular if and only if it is C-regular since $\mathrm{Ass}\,(C) = \mathrm{Ass}\,(R)$. Now an easy induction yields the equality. □

Definition 2.5. Let C be a semidualizing R-module. A finitely generated R-module M is said to be *totally C-reflexive* if the following conditions are satisfied:

(i) The natural evaluation map $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism.

(ii) $\mathrm{Ext}^i_R(M, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C)$ for all $i > 0$.

For an R-module M, if there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$, of R-modules such that each G_i is totally C-reflexive, then we say that M has G_C-dimension at most n, and write $\mathrm{G}_C\text{-}\dim_R(M) \leq n$. If there is no shorter such sequence, we set $\mathrm{G}_C\text{-}\dim_R(M) = n$. Also, if such an integer n does not exist, then we say that M has infinite G_C-dimension, and write $\mathrm{G}_C\text{-}\dim_R(M) = \infty$.

The next proposition collects basic properties of G_C-dimension. For the proof, see [10].

Proposition 2.6. *Let $(R, \mathfrak{m})$ be local, M a finitely generated R-module and let C be a semidualizing R-module. The following statements hold:*

(i) *If $\mathrm{G}_C\text{-}\dim_R(M) < \infty$, and $x \in \mathfrak{m}$ is M-regular, then*
$$\mathrm{G}_C\text{-}\dim_R(M) = \mathrm{G}_C\text{-}\dim_R(M/xM) - 1.$$
If, also, x is R-regular, then
$$\mathrm{G}_C\text{-}\dim_R(M) = \mathrm{G}_{C/xC}\text{-}\dim_{R/xR}(M/xM).$$

(ii) *If $\mathrm{G}_C\text{-}\dim_R(M) < \infty$ and x is an R-regular element in $\mathrm{Ann}_R(M)$, then*
$$\mathrm{G}_C\text{-}\dim_R(M) = \mathrm{G}_{C/xC}\text{-}\dim_{R/xR}(M) + 1.$$

(iii) *Let $0 \to K \to L \to N \to 0$ be a short exact sequence of R-modules. If two of L, K, N are of finite G_C-dimension, then so is the third.*

(iv) *If $\mathrm{G}_C\text{-}\dim_R(M) < \infty$, then*
$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R(M) &= \sup\{i \geq 0 \mid \mathrm{Ext}^i_R(M, C) \neq 0\}\\ &= \mathrm{depth}\,(R) - \mathrm{depth}_R(M).\end{aligned}$$

Definition 2.7. Let C be a semidualizing R-module. The *Auslander class with respect to C* is the class $\mathcal{A}_C(R)$ of R-modules M such that:

(i) $\mathrm{Tor}^R_i(C, M) = 0 = \mathrm{Ext}^i_R(C, C \otimes_R M)$ for all $i \geq 1$, and

(ii) The natural map $M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class with respect to* C is the class $\mathcal{B}_C(R)$ of R-modules M such that:

(i) $\operatorname{Ext}^i_R(C, M) = 0 = \operatorname{Tor}^R_i(C, \operatorname{Hom}_R(C, M))$ for all $i \geq 1$, and

(ii) The natural map $C \otimes_R \operatorname{Hom}_R(C, M)) \to M$ is an isomorphism.

The class $\mathcal{A}_C(R)$ contains all R-modules of finite projective dimension and those of finite C-injective dimension. Also the class $\mathcal{B}_C(R)$ contains all R-modules of finite injective dimension and those of finite C-projective dimension (see [18, Corollary 2.9]). Also, if any two R-modules in a short exact sequence are in $\mathcal{A}_C(R)$ (resp. $\mathcal{B}_C(R)$), then so is the third (see [13]).

Proposition 2.8. *Let* $(R, \mathfrak{m})$ *be a local ring and let* C *be a semidualizing* R*-module.*

(i) C *is a dualizing* R*-module if and only if* $C \otimes_R \widehat{R}$ *is a dualizing* $\widehat{R}$*-module .*

(ii) *Let* $x \in \mathfrak{m}$ *be* R*-regular. Then* C *is a dualizing* R*-module if and only if* C/xC *is a dualizing* R/xR*-module.*

Proof. Just use the definition of dualizing modules. □

Theorem 2.9. *Let* C *be a semidualizing* R*-module and let* M *be an* R*-module.*

(i) $C\text{-id}_R(M) = \operatorname{id}_R(C \otimes_R M)$ *and* $\operatorname{id}_R(M) = C\text{-id}_R(\operatorname{Hom}_R(C, M))$.

(ii) $C\text{-fd}_R(M) = \operatorname{fd}_R(\operatorname{Hom}_R(C, M))$ *and* $\operatorname{fd}_R(M) = C\text{-fd}_R(C \otimes_R M)$.

Proof. For (i), see [18, Theorem 2.11] and for (ii), see [19, Proposition 5.2]. □

Lemma 2.10. *Let* C *be a semidualizing* R*-module,* E *be an injective cogenerator and* M *be an* R*-module.*

(i) *One has* $C\text{-id}_R(M) = C\text{-fd}_R(\operatorname{Hom}_R(M, E))$.

(ii) *One has* $C\text{-fd}_R(M) = C\text{-id}_R(\operatorname{Hom}_R(M, E))$.

Proof. (i). We have the following equalities

$$\begin{aligned} C\text{-id}_R(M) &= \operatorname{id}_R(C \otimes_R M) \\ &= \operatorname{fd}_R(\operatorname{Hom}_R(C \otimes_R M, E)) \\ &= \operatorname{fd}_R(\operatorname{Hom}_R(C, \operatorname{Hom}_R(M, E)) \\ &= C\text{-fd}_R(\operatorname{Hom}_R(M, E)), \end{aligned}$$

in which the first equality is from Theorem 2.9(i), and the last one is from Theorem 2.9(ii).

(ii). Is similar to (i). □

Remark 2.11. Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module. We use $\nu_R(M)$ to denote the minimal number of generators of M. More precisely, $\nu_R(M) = \operatorname{vdim}_{R/\mathfrak{m}}(M \otimes_R R/\mathfrak{m})$. It is easy to see that if $x \in \mathfrak{m}$, then $\nu_R(M) = \nu_{R/xR}(M/xM)$. In particular, if $x \in \operatorname{Ann}_R(M)$, then $\nu_R(M) = \nu_{R/xR}(M)$. Assume that $\operatorname{depth}_R(M) = n$. The type of M, denoted by $r_R(M)$, is defined to be $\operatorname{vdim}_{R/\mathfrak{m}}(\operatorname{Ext}^n_R(R/\mathfrak{m}, M))$. If $x \in \mathfrak{m}$, then $r_R(M/xM) = r_{R/xR}(M/xM)$ by [2, Exercise 1.2.26]. Also, if $x \in \mathfrak{m}$ is M- and R-regular, then $r_R(M) = r_{R/xR}(M/xM)$ by [2, Lemma 3.1.16]. Assuma that C is a semidualizing

R-module. Then $r_R(C) \mid r_R(R)$. Indeed, by reduction modulo a maximal R-sequence, we can assume that $\operatorname{depth}_R(C) = 0 = \operatorname{depth}(R)$. Then we have

$$\begin{aligned} r_R(R) &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m}, R) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Hom}_R(C, C)) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R C, C) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R C \otimes_{R/\mathfrak{m}} R/\mathfrak{m}, C) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_{R/\mathfrak{m}}(R/\mathfrak{m} \otimes_R C, \operatorname{Hom}_R(R/\mathfrak{m}, C)) \\ &= \nu_R(C) r_R(C). \end{aligned}$$

In particular, if $r_R(R) = 1$ (e.g. R is Gorenstein local), then $\nu_R(C) = 1$ and then $C \cong R$.

Definition 2.12. Let M be an R-module and let $\mathcal{X}$ be a class of R-modules . Following [7], a *$\mathcal{X}$-precover* of M is a homomorphism $\varphi : X \to M$, with $X \in \mathcal{X}$, such that every homomorphism $Y \to M$ with $Y \in \mathcal{X}$, factors through ϕ; i.e., the homomorphism

$$\operatorname{Hom}_R(Y, \varphi) : \operatorname{Hom}_R(Y, X) \to \operatorname{Hom}_R(Y, M)$$

is surjective for each module Y in $\mathcal{X}$. A $\mathcal{X}$-precover $\varphi : X \to M$ is a *$\mathcal{X}$-cover* if every $\psi \in \operatorname{Hom}_R(X, X)$ with $\varphi\psi = \varphi$ is an automorphism.

Definition 2.13. Following [6], an R-module M is called *cotorsion* if $\operatorname{Ext}^1_R(F, M) = 0$ for any flat R-module F.

Remark 2.14. In [1], E. Enochs et al. showed that if R is any ring, then every R-module has a flat cover. It is easy to see that flat cover must be surjective. By [6, Lemma 2.2], the kernel of a flat cover is always cotorsion. So that if $F \to M$ is flat cover and M is cotorsion, then so is F. Therefore for an R-module M, one can iteratively take flat covers to construct a flat resolution of M. Since flat cover is unique up to isomorphism, this resolution is unique up to isomorphism of complexes. Such a resolution is called the minimal flat resolution of M. Note that the minimal flat resolution of M is a direct summand of any other flat resolution of M. Assume that

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0,$$

is the minimal flat resolution of M. Then F_i is cotorsion for all $i \geq 1$. If, in addition, M is cotorsion, then all the flat modules in the minimal flat resolution of M are cotorsion. E. Enochs [6], determined the structure of flat cotorsion modules. He showed that if F is flat and cotorsion, then $F \cong \prod\limits_{\mathfrak{p}} T_{\mathfrak{p}}$ where $T_{\mathfrak{p}}$ is the completion of a free $R_{\mathfrak{p}}$-module with respect to $\mathfrak{p}R_{\mathfrak{p}}$-adic topology. So that we can determine the structure of the minimal flat resolution of cotorsion modules.

Definition 2.15. Let M be a cotorsion R-module and let

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0,$$

be the minimal flat resolution of M. Following [8], for a prime ideal $\mathfrak{p}$ of R and an integer $i \geq 0$, the invariant $\pi_i(\mathfrak{p}, M)$ is defined to be the cardinality of the basis of a free $R_{\mathfrak{p}}$-module whose completion is $T_{\mathfrak{p}}$ in the product $F_i \cong \prod\limits_{\mathfrak{p}} T_{\mathfrak{p}}$. By [8, theorem 2.2], for each $i \geq 0$,

$$\pi_i(\mathfrak{p}, M) = \operatorname{vdim}_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}} \operatorname{Tor}_i^{R_{\mathfrak{p}}}\big(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, \operatorname{Hom}_R(R_{\mathfrak{p}}, M)\big).$$

Remark 2.16. Let M be a finitely generated R-module. There are isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(M, E(R/\mathfrak{p})) &\cong \operatorname{Hom}_R(M, E(R/\mathfrak{p}) \otimes_R R_\mathfrak{p}) \\ &\cong \operatorname{Hom}_R(M, E(R/\mathfrak{p})) \otimes_R R_\mathfrak{p} \\ &\cong \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})),\end{aligned}$$

where the the first isomorphism holds because $E(R/\mathfrak{p}) \cong E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$, and the second isomorphism is tensor-evaluation [7, Theorem 3.2.14].

3. Finiteness of G_C-dimension

Throughout this section, C is a semidualizing R-module. We begin with three lemmas that are needed for the main result of this section. It is well-known that a local ring over which there exists a non-zero finitely generated injective module, must be Artinian. Our first lemma generalizes this fact by replacing the injectivity condition with weaker assumption.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be local and let M be a finitely generated R-module with* $\operatorname{depth}(M) = 0$*. If $\operatorname{Ext}^1_R(R/\mathfrak{m}, M) = 0$, then R is Artinian. In particular, M is injective.*

Proof. We show that $\dim(R) = 0$. Assume, on the contrary, that $\dim(R) > 0$. Note that if N is an R-module of finite length, then by using a composition series for N in conjunction with the assumption, we have $\operatorname{Ext}^1_R(N, M) = 0$. Now an easy induction on $\ell_R(N)$ yields the equality $\ell_R(\operatorname{Hom}_R(N, M)) = \ell_R(N)\ell_R(\operatorname{Hom}_R(R/\mathfrak{m}, M))$. Next, note that $\ell_R(R/\mathfrak{m}^i) < \infty$ for any $i \geq 1$, and that the sequence $\{\ell_R(R/\mathfrak{m}^i)\}_{i=1}^\infty$ is not bounded since $\mathfrak{m}^i \neq \mathfrak{m}^{i+1}$ for any $i \geq 1$. Hence $\{\ell_R(\operatorname{Hom}_R(R/\mathfrak{m}^i, M))\}_{i=1}^\infty$ is not bounded. But $0 :_M \mathfrak{m} \subseteq 0 :_M \mathfrak{m}^2 \subseteq \cdots$ is a chain of submodules of M, and hence is eventually stationary. This is a contradiction. Therefore R is Artinian. Finally, the assumption $\operatorname{Ext}^1_R(R/\mathfrak{m}, M) = 0$ implies that M is injective. $\square$

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a local ring and let M be a Cohen-Macaulay R-module with G_C-$\dim_R(M) < \infty$. Then $r_R(C) \mid r_R(M)$.*

Proof. We use induction on $n = \operatorname{depth}(R)$. If $n = 0$, then by Proposition 2.6(iv), we have G_C-$\dim_R(M) = 0$, and hence there is an isomorphism $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$, and the equalities $\operatorname{depth}_R(C) = 0 = \operatorname{depth}_R(M)$. Hence we have

$$\begin{aligned} r_R(M) &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m}, M) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R \operatorname{Hom}_R(M, C), C) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R \operatorname{Hom}_R(M, C) \otimes_{R/\mathfrak{m}} R/\mathfrak{m}, C) \\ &= \operatorname{vdim}_{R/\mathfrak{m}} \operatorname{Hom}_{R/\mathfrak{m}}(R/\mathfrak{m} \otimes_R \operatorname{Hom}_R(M, C), \operatorname{Hom}_R(R/\mathfrak{m}, C)) \\ &= \nu_R(\operatorname{Hom}_R(M, C)) r_R(C). \end{aligned}$$

Therefore $r_R(C) \mid r_R(M)$. Now, assume inductively that $n > 0$. We consider two cases:

Case 1. If $\operatorname{depth}_R(M) = 0$, then M is of finite length since it is Cohen-Macaulay. Hence we can take an R-regular element x such that $xM = 0$. Set $\overline{(-)} = (-) \otimes_R R/xR$. Then by

Proposition 2.6(ii), we have $\mathrm{G}_{\overline{C}}\text{-}\dim_{\overline{R}}(M) < \infty$. Also, note that M is a Cohen-Macaulay $\overline{R}$-module. Hence by induction hypothesis we have $r_{\overline{R}}(\overline{C}) \mid r_{\overline{R}}(M)$. Thus $r_R(C) \mid r_R(M)$.

Case 2. If $\operatorname{depth}_R(M) > 0$, then we can take an element $y \in \mathfrak{m}$ to be M- and R-regular. Set $\overline{(-)} = (-) \otimes_R R/yR$. Now $\overline{M}$ is a Cohen-Macaulay $\overline{R}$-module, and that

$$\mathrm{G}_{\overline{C}}\text{-}\dim_{\overline{R}}(\overline{M}) = \mathrm{G}_C\text{-}\dim_R(M) < \infty,$$

by Proposition 2.6(i). Therefore, by induction hypothesis, we have $r_{\overline{R}}(\overline{C}) \mid r_{\overline{R}}(\overline{M})$, whence $r_R(C) \mid r_R(M)$. This complete the inductive step. □

Lemma 3.3. *Let $(R, \mathfrak{m})$ be local and that $r_R(C) = 1$. If there exists a totally C-reflexive R-module of finite length, then C is dualizing.*

Proof. Assume that M is a finite length C-reflexive R-module. Then $\operatorname{depth}_R(M) = 0$, and hence $\operatorname{depth}(R) = \mathrm{G}_C\text{-}\dim(M) = 0$ by Proposition 2.6(iv). Therefore, we have $\operatorname{depth}_R(C) = 0$ by Proposition 2.4(iv). Now assume, on the contrary, that C is not dualizing. Hence, by Lemma 3.1, we have $\operatorname{Ext}^1_R(R/\mathfrak{m}, C) \neq 0$. Let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,$$

be a composition series for M. Thus the factors are all isomorphic to $R/\mathfrak{m}$, and we have exact sequences

$$0 \to M_{i-1} \to M_i \to R/\mathfrak{m} \to 0,$$

for all $1 \leq i \leq r$. Applying the functor $\operatorname{Hom}_R(-, C)$, we get the exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{m}, C) \to \operatorname{Hom}_R(M_i, C) \to \operatorname{Hom}_R(M_{i-1}, C),$$

for each $1 \leq i \leq r-1$. Now since $\operatorname{depth}_R(C) = 0$ and $r_R(C) = 1$, we have $\operatorname{Hom}_R(R/\mathfrak{m}, C) \cong R/\mathfrak{m}$. Hence we have the inequality $\ell_R(\operatorname{Hom}_R(M_i, C)) \leq \ell_R(\operatorname{Hom}_R(M_{i-1}, C)) + 1$ for each $1 \leq i \leq r - 1$. On the other hand, application of the functor $\operatorname{Hom}_R(-, C)$ on the exact sequence $0 \to M_{r-1} \to M \to R/\mathfrak{m} \to 0$, yields an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_R(R/\mathfrak{m}, C) \to \operatorname{Hom}_R(M, C) \to \operatorname{Hom}_R(M_{r-1}, C) \\ \to \operatorname{Ext}^1_R(R/\mathfrak{m}, C) \to \operatorname{Ext}^1_R(M, C) = 0. \end{aligned}$$

Therefore $\ell_R(\operatorname{Hom}_R(M, C)) = \ell_R(\operatorname{Hom}_R(M_{r-1}, C)) + 1 - \ell_R(\operatorname{Ext}^1_R(R/\mathfrak{m}, C))$. But since $\ell_R(\operatorname{Ext}^1_R(R/\mathfrak{m}, C)) > 0$, we have

$$\begin{aligned} \ell_R(\operatorname{Hom}_R(M, C)) &< \ell_R(\operatorname{Hom}_R(M_{r-1}, C)) + 1 \\ &\leq \ell_R(\operatorname{Hom}_R(M_{r-2}, C)) + 2 \\ &\leq \cdots \\ &\leq \ell_R(\operatorname{Hom}_R(M_0, C)) + r \\ &= r \\ &= \ell_R(M). \end{aligned}$$

Now since $\operatorname{Hom}_R(M, C)$ is again a totally C-reflexive R-module of finite length, the same argument shows that $\ell_R\big(\operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)\big) \leq \ell_R(\operatorname{Hom}_R(M, C))$. But since M is totally C-reflexive, we have $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$, which implies that $\ell_R(M) < \ell_R(M)$, a contradiction. Hence C is dualizing. □

The following theorem is a generalization of [17, Theorem 2.3].

Theorem 3.4. *Let $(R, \mathfrak{m})$ be local. The following are equivalent:*

(i) *C is dualizing.*

(ii) *There exists an ideal $\mathfrak{a}$ with $\mathrm{G}_C\text{-}\dim_R(\mathfrak{a}C) < \infty$ such that $C/\mathfrak{a}C$ is dualizing for $R/\mathfrak{a}$.*

(iii) *There exists a Cohen-Macaulay R-module M with $r_R(M) = 1$ and $\mathrm{G}_C\text{-}\dim_R(M) < \infty$.*

(iv) *$r_R(C) = 1$ and there exists a Cohen-Macaulay R-module M of finite G_C-dimension.*

Proof. (i)$\Longrightarrow$(ii). Choose $\mathfrak{a} = 0$.

(ii)$\Longrightarrow$(iii). We show that $C/\mathfrak{a}C$ has the desired properties. First, the exact sequence

$$0 \to \mathfrak{a}C \to C \to C/\mathfrak{a}C \to 0,$$

in conjunction with Proposition 2.6(iii), show that $\mathrm{G}_C\text{-}\dim_R(C/\mathfrak{a}C) < \infty$. On the other hand, $C/\mathfrak{a}C$ is a Cohen-Macaulay $R/\mathfrak{a}R$-module and hence is a Cohen-Macaulay R-module. Finally, by [2, Exercise 1.2.26], we have $r_R(C/\mathfrak{a}C) = r_{R/\mathfrak{a}}(C/\mathfrak{a}C) = 1$.

(iii)$\Longrightarrow$(iv). By Lemma 3.2, we have $r_R(C) = 1$.

(iv)$\Longrightarrow$(i). Assume that M is a Cohen-Macaulay R-module with $\mathrm{G}_C\text{-}\dim_R(M) < \infty$. We use induction on $m = \operatorname{depth}_R(M)$. If $m = 0$, then M is of finite length since it is Cohen-Macaulay. Since $\sqrt{\operatorname{Ann}_R(M)} = \mathfrak{m}$, we can choose a maximal R-sequence from elements of $\operatorname{Ann}_R(M)$, say $\mathbf{x}$. In view of Proposition 2.8(ii) and Proposition 2.6(ii), we can replace C by $C/\mathbf{x}C$ and R by $R/\mathbf{x}R$, and assume that M is totally C-reflexive. In this case, C is dualizing by Lemma 3.3. Now assume inductively that $m > 0$. Hence $\operatorname{depth}(R) > 0$ by Proposition 2.6(iv), and we can take an element $x \in \mathfrak{m}$ to be M- and R-regular. Set $\overline{(-)} = (-) \otimes_R R/xR$. Now $\overline{M}$ is a Cohen-Macaulay $\overline{R}$-module and $r_{\overline{R}}(\overline{C}) = r_R(C) = 1$. Also, by Proposition 2.6(i), we have $\mathrm{G}_{\overline{C}}\text{-}\dim_{\overline{R}}(\overline{M}) = \mathrm{G}_C\text{-}\dim_R(M) < \infty$. Hence, by induction hypothesis, $\overline{C}$ is dualizing for $\overline{R}$, whence C is dualizing for R by Proposition 2.8(ii). $\square$

It is well-known that the existence of a finitely generated (resp. Cohen-Macaulay) module of finite injective (resp. projective) dimension implies Cohen-Macaulyness of the ring. But, in the special case that C is dualizing, the proof is easy, as the following relations show

$$\dim(R) = \dim_R(C) \leq \operatorname{id}_R(C) = \operatorname{depth}(R),$$

where the first equality is from Proposition 2.4(i), and the remaining parts are from [2, Theorem 3.1.17]. Therefore, in view of Theorem 3.4, we can state the following corollary.

Corollary 3.5. *Let $(R, \mathfrak{m})$ be local. If there exists a Cohen-Macaulay R-module of type 1 and of finite G_C-dimension, then R is Cohen-Macaulay.*

4. C-injective modules

In this section, our aim is to extend two nice results of J.Xu [22]. It is well-known that a Noetherian ring R is Gorenstein if and only if $\mu^i(\mathfrak{p}, R) = \delta_{i,ht(\mathfrak{p})}$ (the Kronecker δ). As a generalization, J.Xu [22, Theorem 2.1], showed that R is Gorenstein if and only if for any R-module F, the necessary and sufficient condition for F to be flat is that $\mu^i(\mathfrak{p}, F) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $i \neq \operatorname{ht}(\mathfrak{p})$. Next, in [22, Theorem 3.2], he proved a dual for this theorem. Indeed, he proved that R is Gorenstein if and only if for any R-module E, the necessary and sufficient condition for E to be injective is that $\pi_i(\mathfrak{p}, E) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and

all $i \neq \mathrm{ht}\,(\mathfrak{p})$. In the present section, first we generalize the mentioned results. Next, we use our new results to determine the minimal flat resolution of some top local cohomology of a Cohen-Macaulay local rings and their torsion products.

Lemma 4.1. *The followings are equivalent:*

(i) *C is pointwise dualizing.*
(ii) *C-$\mathrm{fd}_R(E(R/\mathfrak{m})) = \mathrm{ht}\,(\mathfrak{m})$ for any $\mathfrak{m} \in \mathrm{Max}\,(R)$.*
(iii) *C-$\mathrm{fd}_R(E(R/\mathfrak{m})) < \infty$ for any $\mathfrak{m} \in \mathrm{Max}\,(R)$.*
(iv) *C-$\mathrm{fd}_R(E(R/\mathfrak{p})) = \mathrm{ht}\,(\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*
(v) *C-$\mathrm{fd}_R(E(R/\mathfrak{p})) < \infty$ for any $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*
(vi) *C-$\mathrm{id}_R(T_{\mathfrak{m}}) = \mathrm{ht}\,(\mathfrak{m})$ for any $\mathfrak{m} \in \mathrm{Max}\,(R)$.*
(vii) *C-$\mathrm{id}_R(T_{\mathfrak{m}}) < \infty$ for any $\mathfrak{m} \in \mathrm{Max}\,(R)$.*
(viii) *C-$\mathrm{id}_R(T_{\mathfrak{p}}) = \mathrm{ht}\,(\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*
(ix) *C-$\mathrm{id}_R(T_{\mathfrak{p}}) < \infty$ for any $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*

Proof. (i)$\Longrightarrow$(ii). Assume that $\mathfrak{m} \in \mathrm{Max}\,(R)$. There are equalities

$$\begin{aligned} C\text{-}\mathrm{fd}_R(E(R/\mathfrak{m})) &= \mathrm{fd}_R(\mathrm{Hom}_R(C, E(R/\mathfrak{m}))) \\ &= \mathrm{fd}_{R_{\mathfrak{m}}}(\mathrm{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, E_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}})) \\ &= \mathrm{id}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}) \\ &= \dim\,(R_{\mathfrak{m}}) \\ &= \mathrm{ht}\,(\mathfrak{m}), \end{aligned}$$

in which the first equality is from Theorem 2.9(ii), and the second one is from Remark 2.16.

(ii)$\Longrightarrow$(iii). Is clear.

(iii)$\Longrightarrow$(i). We can assume that $(R, \mathfrak{m})$ is local. Now one can use Theorem 2.9(ii), to see that

$$\begin{aligned} \mathrm{id}_R(C) &= \mathrm{fd}_R(\mathrm{Hom}_R(C, E(R/\mathfrak{m}))) \\ &= C\text{-}\mathrm{fd}_R(E(R/\mathfrak{m})) < \infty, \end{aligned}$$

whence C is dualizing.

(i)$\Longrightarrow$(iv). Let $\mathfrak{p}$ be a prime ideal of R. Note that $E(R/\mathfrak{p})_{\mathfrak{q}} \neq 0$ if and only if $\mathfrak{q} \subseteq \mathfrak{p}$. Now as in (i)$\Longrightarrow$(ii), we have C-$\mathrm{fd}_R(E(R/\mathfrak{p})) = \dim\,(R_{\mathfrak{p}}) = \mathrm{ht}\,(\mathfrak{p})$.

(iv)$\Longrightarrow$(v). Is clear.

(v)$\Longrightarrow$(i). Again, we can assume that R is local. Now the proof is similar to that of (iii)$\Longrightarrow$(i).

(ii)$\Longleftrightarrow$(vi) and (iii)$\Longleftrightarrow$(vii). Note that $T_{\mathfrak{m}} = \mathrm{Hom}_R\big(E(R/\mathfrak{m}), E(R/\mathfrak{m})^{(X)}\big)$ for some set X. Now we have the equalities

$$\begin{aligned} C\text{-}\mathrm{id}_R(T_{\mathfrak{m}}) &= \mathrm{id}_R(C \otimes_R T_{\mathfrak{m}}) \\ &= \mathrm{id}_R\big(C \otimes_R \mathrm{Hom}_R\big(E(R/\mathfrak{m}), E(R/\mathfrak{m})^{(X)}\big)\big) \\ &= \mathrm{id}_R\big(\mathrm{Hom}_R\big(\mathrm{Hom}_R(C, E(R/\mathfrak{m})), E(R/\mathfrak{m})^{(X)}\big)\big) \\ &= \mathrm{fd}_R(\mathrm{Hom}_R(C, E(R/\mathfrak{m}))) \\ &= C\text{-}\mathrm{fd}_R(E(R/\mathfrak{m})), \end{aligned}$$

in which the first equality is from Theorem 2.9(i), the fourth equality is from Remark 2.16 and the fact that $E(R/\mathfrak{m})^{(X)}$ is an injective cogenerator in the category of $R_\mathfrak{m}$-modules, and the last one is from Theorem 2.9(ii).

(iv)$\Longleftrightarrow$(viii) and (v)$\Longleftrightarrow$(ix). Are similar to (ii)$\Longleftrightarrow$(vi). □

The following theorem is a generalization of [21, theorem 2.1].

Theorem 4.2. *The following are equivalent:*

(i) *C is pointwise dualizing.*
(ii) *An R-module M is C-flat if and only if $\mu^i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq \operatorname{ht}(\mathfrak{p})$.*
(iii) *An R-module M is flat if and only if $\mu^i(\mathfrak{p}, C \otimes_R M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq \operatorname{ht}(\mathfrak{p})$.*

Proof. (i)$\Longrightarrow$(ii). First assume that M is C-flat. Set $M = C \otimes_R F$, where F is a flat R-module. Since C is pointwise dualizing, we have $\mu^i(\mathfrak{p}, C) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $i \neq \operatorname{ht}(\mathfrak{p})$. Assume that

$$0 \to C \to E^0(C) \to E^1(C) \to ... \to E^i(C) \to ...$$

is the minimal injective resolution of C. By applying the exact functor $- \otimes_R F$ to this resolution, we find an exact complex

$$0 \to M = C \otimes_R F \to E^0(C) \otimes_R F \to E^1(C) \otimes_R F \to ... \to E^i(C) \otimes_R F \to ..., (*)$$

which is an injective resolution for M. By [7, Theorem 3.3.12] the injective R-module $E(R/\mathfrak{p}) \otimes_R F$ is a direct sum of copies of $E(R/\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Now, since the minimal injective resolution of M is a direct summand of the complex $(*)$, we get the result. Conversely, suppose that M is an R-module such that $\mu^i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq \operatorname{ht}(\mathfrak{p})$. In order to show that M is C-flat, it is enough to prove that $M_\mathfrak{m}$ is $C_\mathfrak{m}$-flat $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \operatorname{Max}(R)$. For if $M_\mathfrak{m}$ is $C_\mathfrak{m}$-flat $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \operatorname{Max}(R)$, then $\operatorname{Hom}_R(C, M)_\mathfrak{m} \cong \operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, M_\mathfrak{m})$ is flat as an $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \operatorname{Max}(R)$ by Theorem 2.9(ii). Hence $\operatorname{Hom}_R(C, M)$ is a flat R-module and thus M is C-flat by Theorem 2.9(ii). Hence, replacing R by $R_\mathfrak{m}$, we can assume that $(R, \mathfrak{m})$ is local. Clearly we may assume that $M \neq 0$. In this case we have $\operatorname{id}_R(M) < \infty$ since by assumption $\mu^i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $i > \dim(R)$. Hence the assumption in conjunction with Lemma 4.1, imply that M has a bounded injective resolution all of whose terms have finite C-flat dimensions. More precisely, by Lemma 4.1, if E^i is the i-th term in the minimal injective resolution of M, then $C\text{-fd}_R(E^i) = i$ for all $0 \leq i \leq \operatorname{id}(M)$. Breaking up this resolution to short exact sequences and using [19, Corollary 5.7], we can conclude that $C\text{-fd}_R(M) = 0$. Hence M is C-flat, as wanted.

(ii)$\Longrightarrow$(iii). Assume that M is a flat R-module. Then $C \otimes_R M \in \mathcal{F}_C$ and $\mu^i(\mathfrak{p}, C \otimes_R M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq \operatorname{ht}(\mathfrak{p})$ by assumption. Conversely, suppose that $\mu^i(\mathfrak{p}, C \otimes_R M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq \operatorname{ht}(\mathfrak{p})$. Then, by assumption, $C \otimes_R M$ is C-flat. Set $C \otimes_R M = C \otimes_R F$, where F is flat. Therefore $C \otimes_R M \in \mathcal{B}_C(R)$, whence

$M \in \mathcal{A}_C(R)$ by [18, Theorem 2.8(b)]. Thus we have the isomorphisms

$$\begin{aligned} M &\cong \operatorname{Hom}_R(C, C \otimes_R M) \\ &\cong \operatorname{Hom}_R(C, C \otimes_R F) \\ &\cong F, \end{aligned}$$

where the first and the last isomorphism hold since both M and F are in $\mathcal{A}_C(R)$.

(iii)$\Longrightarrow$(i). Note that R is a flat R-module. Hence by assumption, if $\mathfrak{m} \in \operatorname{Max}(R)$, then $\mu^i(\mathfrak{m}, C \otimes_R R) = 0$ for all $i > \operatorname{ht}(\mathfrak{m})$. Thus $\operatorname{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) < \infty$, as wanted. □

Theorem 4.3. *The following are equivalent:*

(i) *C is pointwise dualizing.*

(ii) *An R-module M is C-injective if and only if $\pi_i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq ht(\mathfrak{p})$.*

(iii) *An R-module M is injective if and only if $\pi_i(\mathfrak{p}, \operatorname{Hom}_R(C, M)) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq ht(\mathfrak{p})$.*

Proof. (i)$\Longrightarrow$(ii) . Assume that M is a nonzero C-injective R-module. Set $M = \operatorname{Hom}_R(C, E)$ with E is injective. First, we show that M is cotorsion. Assume that F is a flat R-module. Then, by [7, Theorem 3.2.1], we have $\operatorname{Ext}^1_R(F, \operatorname{Hom}_R(C, E)) \cong \operatorname{Hom}_R(\operatorname{Tor}^R_1(F, C), E) = 0$, and hence M is cotorsion. Fix a prime ideal $\mathfrak{p}$ of R and set $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Note that $\operatorname{Hom}_R(R_\mathfrak{p}, E)$ is an injective R-module and that $\operatorname{Hom}_R(R_\mathfrak{p}, E) \cong \underset{\mathfrak{q} \in X}{\oplus} E(R/\mathfrak{q})$, where $X \subseteq \operatorname{Ass}_R(E)$ and each element of X is a subset of $\mathfrak{p}$. There are isomorphisms

$$\begin{aligned} \operatorname{Tor}^{R_\mathfrak{p}}_i\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, \operatorname{Hom}_R(C, E))\big) &\cong \operatorname{Tor}^{R_\mathfrak{p}}_i\big(k(\mathfrak{p}), \operatorname{Hom}_R(C_\mathfrak{p}, E)\big) \\ &\cong \operatorname{Tor}^{R_\mathfrak{p}}_i\big(k(\mathfrak{p}), \operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, \operatorname{Hom}_R(R_\mathfrak{p}, E)\big) \\ &\cong \operatorname{Tor}^{R_\mathfrak{p}}_i\big(k(\mathfrak{p}), \operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, \underset{\mathfrak{q} \in X}{\oplus} E(R/\mathfrak{q})\big) \\ &\cong \operatorname{Hom}_{R_\mathfrak{p}}\big(\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), C_\mathfrak{p}), \underset{\mathfrak{q} \in X}{\oplus} E(R/\mathfrak{q})\big), \end{aligned}$$

where the last isomorphism is from [7, Theorem 3.2.13]. Now since $C_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$, we have $\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), C_\mathfrak{p}) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{p})$. Therefore $\pi_i(\mathfrak{p}, M) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{p})$. Conversely, assume that M is a non-zero R-module with $\pi_i(\mathfrak{p}, M) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{p})$. By assumption, the minimal flat resolution of M is of the form

$$\cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

in which $F_i = \prod\limits_{\operatorname{ht}(\mathfrak{p})=i} T_\mathfrak{p}$ for all $i \geq 1$. Also, in view of [22, Lemma 3.1], we have $F_0 = \prod\limits_{\operatorname{ht}(\mathfrak{p})=0} T_\mathfrak{p}$. Hence the minimal flat resolution of M is of the form

$$\cdots \longrightarrow \prod_{\operatorname{ht}(\mathfrak{p})=i} T_\mathfrak{p} \longrightarrow \cdots \longrightarrow \prod_{\operatorname{ht}(\mathfrak{p})=1} T_\mathfrak{p} \longrightarrow \prod_{\operatorname{ht}(\mathfrak{p})=0} T_\mathfrak{p} \longrightarrow M \longrightarrow 0. \quad (*)$$

Let E be an injective cogenerator. According to Lemma 2.10(i), it is enough to show that $\operatorname{Hom}_R(M, E)$ is C-flat. In fact, by Theorem 4.2, we need only to show that $\mu^i(\mathfrak{p}, \operatorname{Hom}_R(M, E)) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{p})$ and all $i \geq 0$. Applying the exact functor $\operatorname{Hom}_R(-, E)$ on $(*)$, we get an injective resolution

$$0 \longrightarrow \operatorname{Hom}_R(M, E) \longrightarrow \operatorname{Hom}_R\Big(\prod_{\operatorname{ht}(\mathfrak{p})=0} T_\mathfrak{p}, E\Big) \longrightarrow$$

$$\operatorname{Hom}_R\Big(\prod_{\operatorname{ht}(\mathfrak{p})=1} T_{\mathfrak{p}}, E\Big) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_R\Big(\prod_{\operatorname{ht}(\mathfrak{p})=i} T_{\mathfrak{p}}, E\Big) \longrightarrow \cdots,$$

for $\operatorname{Hom}_R(M, E)$. Note that $\operatorname{Hom}_R\Big(\prod_{\operatorname{ht}(\mathfrak{p})=i} T_{\mathfrak{p}}, E\Big)$ is an injective R-module for all $i \geq 0$. Set $\operatorname{Hom}_R\Big(\prod_{\operatorname{ht}(\mathfrak{p})=i} T_{\mathfrak{p}}, E\Big) \cong \oplus E(R/\mathfrak{q})$. We show that $\operatorname{ht}(\mathfrak{q}) = i$. Since C is pointwise dualizing, by Lemma 4.1, we have $C\text{-fd}_R(E(R/\mathfrak{q})) = \operatorname{ht}(\mathfrak{q})$. On the other hand, we have the equalities

$$\begin{aligned} C\text{-fd}_R(E(R/\mathfrak{q})) &= C\text{-fd}_R(\oplus E(R/\mathfrak{q})) \\ &= C\text{-fd}_R\Big(\operatorname{Hom}_R\Big(\prod_{\operatorname{ht}(\mathfrak{p})=i} T_{\mathfrak{p}}, E\Big)\Big) \\ &= C\text{-id}_R\Big(\prod_{\operatorname{ht}(\mathfrak{p})=i} T_{\mathfrak{p}}\Big) \\ &= i, \end{aligned}$$

in which the third equality is from Lemma 2.10(i), and the last one is from Lemma 4.1. Hence $\mu^i(\mathfrak{p}, \operatorname{Hom}_R(M, E)) = 0$ for all $i \geq 0$ with $i \neq \operatorname{ht}(\mathfrak{p})$, as wanted.

(ii)$\Longrightarrow$(iii). Assume that M is an injective R-module. Then $\operatorname{Hom}_R(C, M) \in \mathcal{I}_C$ and $\mu^i(\mathfrak{p}, \operatorname{Hom}_R(C, M)) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq \operatorname{ht}(\mathfrak{p})$ by assumption. Conversely, suppose that $\mu^i(\mathfrak{p}, \operatorname{Hom}_R(C, M)) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $i \neq \operatorname{ht}(\mathfrak{p})$. Then, by assumption, $\operatorname{Hom}_R(C, M)$ is C-injective. Set $\operatorname{Hom}_R(C, M) = \operatorname{Hom}_R(C, I)$, where I is injective. Therefore $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C(R)$, whence $M \in \mathcal{B}_C(R)$ by [18, Theorem 2.8(a)]. Thus we have the isomorphisms

$$\begin{aligned} M &\cong C \otimes_R \operatorname{Hom}_R(C, M) \\ &\cong C \otimes_R \operatorname{Hom}_R(C, I) \\ &\cong I, \end{aligned}$$

where the first and the last isomorphism hold since both M and I are in $\mathcal{B}_C(R)$.

(iii)$\Longrightarrow$(i). Assume that $\mathfrak{m}$ is a maximal ideal of R. Set $k(\mathfrak{m}) = R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$. Since $E(R/\mathfrak{m})$ is injective, by assumption, we have $\pi_i\big(\mathfrak{m}, \operatorname{Hom}_R(C, E(R/\mathfrak{m}))\big) = 0$ fo all $i \neq \operatorname{ht}(\mathfrak{m})$. On the other hand, there are isomorphisms

$$\begin{aligned} \operatorname{Hom}_{R_{\mathfrak{m}}}\big(\operatorname{Ext}^i_{R_{\mathfrak{m}}}(k(\mathfrak{m}), C_{\mathfrak{m}}), E(k(\mathfrak{m}))\big) &\cong \operatorname{Tor}_i^{R_{\mathfrak{m}}}\big(k(\mathfrak{m}), \operatorname{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, E(k(\mathfrak{m})))\big) \\ &\cong \operatorname{Tor}_i^{R_{\mathfrak{m}}}\big(k(\mathfrak{m}), \operatorname{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}}, E(k(\mathfrak{m})))\big) \\ &\cong \operatorname{Tor}_i^{R_{\mathfrak{m}}}\big(k(\mathfrak{m}), \operatorname{Hom}_R(R_{\mathfrak{m}}, \operatorname{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}}, E(k(\mathfrak{m})))\big) \\ &\cong \operatorname{Tor}_i^{R_{\mathfrak{m}}}\big(k(\mathfrak{m}), \operatorname{Hom}_R(R_{\mathfrak{m}}, \operatorname{Hom}_R(C, E(R/\mathfrak{m})))\big), \end{aligned}$$

where the first isomorphism is from [7, Theorem 3.2.13], and the last one is from Remark 2.16. From this isomorphisms, it follows that $\operatorname{Hom}_{R_{\mathfrak{m}}}\big(\operatorname{Ext}^i_{R_{\mathfrak{m}}}(k(\mathfrak{m}), C_{\mathfrak{m}}), E(k(\mathfrak{m}))\big) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{m})$, from which we conclude that $\operatorname{Ext}^i_{R_{\mathfrak{m}}}(k(\mathfrak{m}), C_{\mathfrak{m}}) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{m})$, since $E(k(\mathfrak{m}))$ is an injective cogenerator in the category of $R_{\mathfrak{m}}$-modules. Thus $C_{\mathfrak{m}}$ is dualizing for $R_{\mathfrak{m}}$, as required. $\square$

Corollary 4.4. *Let C be pointwise dualizing. Then flat cover of any C-injective R-module is C-injective.*

Proof. By Lemma 4.1, $C\text{-id}_R(T_\mathfrak{p}) = 0$ for any prime ideal $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 0$. Hence $T_\mathfrak{p}$ is C-injective for any prime ideal $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 0$. Assume that M is a C-injective R-module. By Theorem 4.3, we have $F(M) = \prod_{\mathrm{ht}(\mathfrak{p})=0} T_\mathfrak{p}$. Now the result follows since the class $\mathcal{I}_C$ closed under arbitrary direct product. □

Corollary 4.5. *The R-module C is pointwise dualizing if and only if for any prime ideal $\mathfrak{p}$ of R,*

$$\pi_i\big(\mathfrak{p}, \mathrm{Hom}_R(C, E(R/\mathfrak{p}))\big) = \begin{cases} 1 & i = \mathrm{ht}(\mathfrak{p}), \\ 0 & i \neq \mathrm{ht}(\mathfrak{p}). \end{cases}$$

Proof. Assume that $\mathfrak{p} \in \mathrm{Spec}(R)$. Set $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. We have the following equalities

$$\begin{aligned} \pi_i\big(\mathfrak{p}, \mathrm{Hom}_R(C, E(R/\mathfrak{p}))\big) &= \mathrm{vdim}_{k(\mathfrak{p})}\mathrm{Tor}_i^{R_\mathfrak{p}}\big(k(\mathfrak{p}), \mathrm{Hom}_R(R_\mathfrak{p}, \mathrm{Hom}_R(C, E(R/\mathfrak{p})))\big) \\ &= \mathrm{vdim}_{k(\mathfrak{p})}\mathrm{Tor}_i^{R_\mathfrak{p}}\big(k(\mathfrak{p}), \mathrm{Hom}_R(C_\mathfrak{p}, \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, E(R/\mathfrak{p})))\big) \\ &= \mathrm{vdim}_{k(\mathfrak{p})}\mathrm{Tor}_i^{R_\mathfrak{p}}\big(k(\mathfrak{p}), \mathrm{Hom}_R(C_\mathfrak{p}, E(R/\mathfrak{p}))\big) \\ &= \mathrm{vdim}_{k(\mathfrak{p})}\mathrm{Hom}_{R_\mathfrak{p}}\big(\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), C_\mathfrak{p}), E(R/\mathfrak{p})\big), \end{aligned}$$

where the second equality is from Remark 2.16, and the last equality is from [7, Theorem 3.2.13]. Now, C is pointwise dualizing if and only if $C_\mathfrak{p}$ is the dualizing module of $R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, and this is the case if and only if

$$\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), C_\mathfrak{p}) \cong \begin{cases} k(\mathfrak{p}) & i = \mathrm{ht}(\mathfrak{p}), \\ 0 & i \neq \mathrm{ht}(\mathfrak{p}). \end{cases}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Thus we are done by the above equalities and the fact that $\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(R/\mathfrak{p})) \cong k(\mathfrak{p})$. □

In the following corollaries, we are concerned with the local cohomology. For an R-module M, the i-th local cohomology module of M with respect to an ideal $\mathfrak{a}$ of R, denoted by $\mathrm{H}^i_\mathfrak{a}(M)$, is defined to be

$$\mathrm{H}^i_\mathfrak{a}(M) = \varinjlim_{n\geq 1}\mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For the basic properties of local cohomology modules, please see the textbook [3].

Corollary 4.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim(R) = d$ possesing a canonical module ω_R. Then $\pi_i\big(\mathfrak{m}, \mathrm{H}^d_\mathfrak{m}(R)\big) = \delta_{i,d}$, and $\pi_i\big(\mathfrak{q}, \mathrm{H}^d_\mathfrak{m}(R)\big) = 0$ for any non-maximal prime ideal $\mathfrak{q}$ whenever $i \neq \mathrm{ht}(\mathfrak{q})$.*

Proof. By [3, Theorem 11.2.8], we have $\mathrm{H}^d_\mathfrak{m}(R) \cong \mathrm{Hom}_R(\omega_R, E(R/\mathfrak{m}))$, and hence $\mathrm{H}^d_\mathfrak{m}(R)$ is ω_R-injective. Assume that $\mathfrak{q}$ is a non-maximal prime ideal of R. Then by the Theorem 4.3, we have $\pi_i\big(\mathfrak{q}, \mathrm{H}^d_\mathfrak{m}(R)\big) = 0$ for all $i \neq \mathrm{ht}(\mathfrak{q})$. Finally, by corollary 4.5, we have $\pi_i\big(\mathfrak{m}, \mathrm{H}^d_\mathfrak{m}(R)\big) = 0$ for all $i \neq d$ and that $\pi_d\big(\mathfrak{m}, \mathrm{H}^d_\mathfrak{m}(R)\big) = 1$, as wanted. □

If $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with $\dim(R) = d$, then by [3, Corollary 6.2.9] the only non-vanishing local cohomology of R with respect to $\mathfrak{m}$ is $\mathrm{H}^d_\mathfrak{m}(R)$. Also, if R admits a

canonical module, then by [7, Proposition 9.5.22], we have $\operatorname{fd}_R(\mathrm{H}^d_{\mathfrak{m}}(R)) = d$. The following corollary describes the structure of the minimal flat resolution of $\mathrm{H}^d_{\mathfrak{m}}(R)$.

Corollary 4.7. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring possessing a canonical module. The minimal flat resolution of $\mathrm{H}^d_{\mathfrak{m}}(R)$ is of the form*

$$0 \longrightarrow \widehat{R_{\mathfrak{m}}} \longrightarrow \cdots \longrightarrow \prod_{\mathrm{ht}\,(\mathfrak{p})=1} T_{\mathfrak{p}} \longrightarrow \prod_{\mathrm{ht}\,(\mathfrak{p})=0} T_{\mathfrak{p}} \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(R) \longrightarrow 0.$$

In the following corollary, we give another proof of [16, Corollary 3.7]. Our approach is direct, and uses the well-known fact that the homology functor Tor can be computed by a flat resolution.

Corollary 4.8. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring. Then*

$$\operatorname{Tor}^R_i(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{H}^d_{\mathfrak{m}}(R)) \cong \begin{cases} \mathrm{H}^d_{\mathfrak{m}}(R) & i = d, \\ 0 & i \neq d. \end{cases}$$

Proof. Note that $\widehat{R}$ is a d-dimensional complete Cohen-Macaulay local ring, and hence admits a canonical module $\omega_{\widehat{R}}$. The R-module $\mathrm{H}^d_{\mathfrak{m}}(R)$ is Artinian by [3, Theorem 7.1.6], and thus naturally has a $\widehat{R}$-module structure by [3, Remark 10.2.9]. Hence $\operatorname{Tor}^R_i(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{H}^d_{\mathfrak{m}}(R))$ is Artinian for all $i \geq 0$ by [14, Corollary 3.2]. Thus there are isomorphisms

$$\begin{aligned} \operatorname{Tor}^R_i(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{H}^d_{\mathfrak{m}}(R)) &\cong \operatorname{Tor}^R_i(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{H}^d_{\mathfrak{m}}(R)) \otimes_R \widehat{R} \\ &\cong \operatorname{Tor}^{\widehat{R}}_i(\mathrm{H}^d_{\mathfrak{m}}(R) \otimes_R \widehat{R}, \mathrm{H}^d_{\mathfrak{m}}(R) \otimes_R \widehat{R}) \\ &\cong \operatorname{Tor}^{\widehat{R}}_i\big(\mathrm{H}^d_{\mathfrak{m}\widehat{R}}(\widehat{R}), \mathrm{H}^d_{\mathfrak{m}\widehat{R}}(\widehat{R})\big), \end{aligned}$$

in which the second isomorphism is from [7, Theorem 2.1.11], and the last one is flat base change [3, Theorem 4.3.2]. Also, we have the isomorphisms

$$\begin{aligned} \mathrm{H}^d_{\mathfrak{m}}(R) &\cong \mathrm{H}^d_{\mathfrak{m}}(R) \otimes_R \widehat{R} \\ &\cong \mathrm{H}^d_{\mathfrak{m}\widehat{R}}(\widehat{R}) \\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\omega_{\widehat{R}}, E_{\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R})\big), \end{aligned}$$

in which the first isomorphism holds because $\mathrm{H}^d_{\mathfrak{m}}(R)$ is Artinian, the second isomorphism is the flat base change, and the last one is local duality [3, Theorem 11.2.8]. Thus $\mathrm{H}^d_{\mathfrak{m}}(R)$ is a $\omega_{\widehat{R}}$-injective $\widehat{R}$-module. Hence, by Corollary 4.7, the minimal flat resolution of $\mathrm{H}^d_{\mathfrak{m}}(R)$, as an $\widehat{R}$-module, is of the form

$$0 \longrightarrow \widehat{\widehat{R}_{\mathfrak{m}\widehat{R}}} \longrightarrow \cdots \longrightarrow \prod_{\mathrm{ht}\,(Q)=1} T_Q \longrightarrow \prod_{\mathrm{ht}\,(Q)=0} T_Q \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(R) \longrightarrow 0,$$

in which T_Q is the completion of a free $\widehat{R}_Q$-module with respect to $Q\widehat{R}_Q$-adic topology, for $Q \in \operatorname{Spec}(\widehat{R})$. Observe that the above resolution is a flat resolution of $\mathrm{H}^d_{\mathfrak{m}}(R)$ as an R-module since the modules in the above resolution are all flat R-modules. Therefore, we can replace R by $\widehat{R}$, and assume that R is complete. So that, the minimal flat resolution of $\mathrm{H}^d_{\mathfrak{m}}(R)$ is of the form

$$0 \longrightarrow \widehat{R_{\mathfrak{m}}} \longrightarrow \cdots \longrightarrow \prod_{\mathrm{ht}\,(\mathfrak{p})=1} T_{\mathfrak{p}} \longrightarrow \prod_{\mathrm{ht}\,(\mathfrak{p})=0} T_{\mathfrak{p}} \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(R) \longrightarrow 0,$$

in which $T_{\mathfrak{p}}$ is the completion of a free $R_{\mathfrak{p}}$-module with respect to $\mathfrak{p}R_{\mathfrak{p}}$-adic topology, for $\mathfrak{p} \in \operatorname{Spec}(R)$. Next, note that for each prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$, we have $\mathrm{H}^d_{\mathfrak{m}}(R) \otimes_R \Big(\prod T_{\mathfrak{p}}\Big) = 0$. Indeed, we can write $\mathrm{H}^d_{\mathfrak{m}}(R) = \varinjlim_{\alpha \in I} M_\alpha$, where M_α is a finitely generated submodule of $\mathrm{H}^d_{\mathfrak{m}}(R)$. Also $T_{\mathfrak{p}} = \operatorname{Hom}_R\big(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X)}\big)$ for some set X. Now since M_α is of finite length by [3, Theorem 7.1.3], we can take an element $x \in \mathfrak{m} \smallsetminus \mathfrak{q}$ such that $xM_\alpha = 0$. But multiplication of x induces an automorphism on $E(R/\mathfrak{p})$ and hence on $\prod T_{\mathfrak{p}}$. Consequently, multiplication of x on $M_\alpha \otimes_R \Big(\prod T_{\mathfrak{p}}\Big)$ is both an isomorphism and zero. Hence $M_\alpha \otimes_R \Big(\prod T_{\mathfrak{p}}\Big) = 0$, from which we conclude that $\mathrm{H}^d_{\mathfrak{m}}(R) \otimes_R \Big(\prod T_{\mathfrak{p}}\Big) = 0$ since tensor commutes with direct limit. Thus $\operatorname{Tor}^R_i(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{H}^d_{\mathfrak{m}}(R)) = 0$ for $i \neq d$. Finally, we have

$$\begin{aligned}
\operatorname{Tor}^R_d(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{H}^d_{\mathfrak{m}}(R)) &\cong \widehat{R_{\mathfrak{m}}} \otimes_R \mathrm{H}^d_{\mathfrak{m}}(R) \\
&\cong \mathrm{H}^d_{\mathfrak{m}\widehat{R_{\mathfrak{m}}}}(\widehat{R_{\mathfrak{m}}}) \\
&\cong \operatorname{Hom}_{\widehat{R_{\mathfrak{m}}}}(\widehat{\omega_{R_{\mathfrak{m}}}}, E_{\widehat{R_{\mathfrak{m}}}}(\widehat{R_{\mathfrak{m}}}/\mathfrak{m}\widehat{R_{\mathfrak{m}}})) \\
&\cong \operatorname{Hom}_{R_{\mathfrak{m}}}(\omega_{R_{\mathfrak{m}}}, E_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}})) \otimes_{R_{\mathfrak{m}}} \widehat{R_{\mathfrak{m}}} \\
&\cong \operatorname{Hom}_{R_{\mathfrak{m}}}(\omega_{R_{\mathfrak{m}}}, E_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}})) \\
&\cong \operatorname{Hom}_R(\omega_R, E(R/\mathfrak{m})) \otimes_R R_{\mathfrak{m}} \\
&\cong \operatorname{Hom}_R(\omega_R, E(R/\mathfrak{m}) \otimes_R R_{\mathfrak{m}}) \\
&\cong \operatorname{Hom}_R(\omega_R, E(R/\mathfrak{m})) \\
&\cong \mathrm{H}^d_{\mathfrak{m}}(R),
\end{aligned}$$

in which the second isomorphism is the flat base change [3, Theorem 4.3.2], the third isomorphism is local duality [3, Theorem 11.2.8], and the fifth one is from [3, Remark 10.2.9], since $\operatorname{Hom}_{R_{\mathfrak{m}}}(\omega_{R_{\mathfrak{m}}}, E_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}))$ is an Artinian $R_{\mathfrak{m}}$-module and hence has a natural structure as an $\widehat{R_{\mathfrak{m}}}$-module. □

The following theorem is a slight generalization of [22, Theorem 3.3].

Theorem 4.9. *The following are equivalent:*

(i) *C is pointwise dualizing.*

(ii) *If M is a cotorsion R-module such that $C\text{-}\mathrm{id}_R(M) = n < \infty$, then M admits a minimal flat resolution such that $\pi_i(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ whenever $\operatorname{ht}(\mathfrak{p}) \notin \{i, ..., i+n\}$.*

Proof. (i) $\Longrightarrow$ (ii). We use induction on n. If $n = 0$, then we are done by Theorem 4.3. Now assume inductively that $n > 0$ and the case n ie settled. Fix a prime ideal $\mathfrak{p}$ of R. Assume that M is a cotorsion R-module with $C\text{-}\mathrm{id}_R(M) = n+1$. Hence $M \in \mathcal{A}_C(R)$, and so the $\mathcal{I}_C$-preenvelope of M is injective by [18, Corollary 2.4(b)]. Thus there exists an exact sequence

$$0 \to M \to \operatorname{Hom}_R(C, I) \to L \to 0, \ (*)$$

in which I is injective, and $L = \operatorname{Coker}(M \to \operatorname{Hom}_R(C, I))$. Note that L is cotorsion since both M and $\operatorname{Hom}_R(C, I)$ are cotorsion. Also, since both M and $\operatorname{Hom}_R(C, I)$ are in $\mathcal{A}_C(R)$, we have $L \in \mathcal{A}_C(R)$, and therefore $\operatorname{Tor}^R_1(C, L) = 0$. On the other hand

$C \otimes_R \operatorname{Hom}_R(C, I) \cong I$, by [7, Theorem 3.2.11]. Hence application of $C \otimes_R -$ on $(*)$ yields an exact sequence

$$0 \to C \otimes_R M \to I \to C \otimes_R L \to 0.$$

By Theorem 2.9(i), we have $\operatorname{id}_R(C \otimes_R M) = n + 1$. Therefore $\operatorname{id}_R(C \otimes_R L) = n$, whence $C\text{-}\operatorname{id}_R(L) = n$. Now induction hypothesis applied to $\operatorname{Hom}_R(C, I)$ and L yields that $\pi_i\big(\mathfrak{p}, \operatorname{Hom}_R(C, I)\big) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{p})$, and that $\pi_i(\mathfrak{p}, L) = 0$ fo all $\operatorname{ht}(\mathfrak{p}) \notin \{i, ..., i+n\}$. Note that $\operatorname{Ext}^1_R(R_\mathfrak{p}, M) = 0$ since M is cotorsion. Hence the exact sequence $(*)$ yields an exact sequence

$$0 \to \operatorname{Hom}_R(R_\mathfrak{p}, M) \to \operatorname{Hom}_R(R_\mathfrak{p}, \operatorname{Hom}_R(C, I)) \to \operatorname{Hom}_R(R_\mathfrak{p}, L) \to 0,$$

and the later exact sequence, by applying $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} -$, yields the long exact sequence

$$\cdots \to \operatorname{Tor}^{R_\mathfrak{p}}_{i+1}\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, \operatorname{Hom}_R(C, E))\big) \to \operatorname{Tor}^{R_\mathfrak{p}}_{i+1}\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, L)\big) \to$$
$$\operatorname{Tor}^{R_\mathfrak{p}}_{i}\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, M)\big) \to \operatorname{Tor}^{R_\mathfrak{p}}_{i}\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, \operatorname{Hom}_R(C, E))\big) \to \cdots.$$

From the above long exact sequence, it follows that $\operatorname{Tor}^{R_\mathfrak{p}}_{i}\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, M)\big) = 0$ for all $\operatorname{ht}(\mathfrak{p}) \notin \{i, ..., i+n+1\}$, as wanted. This completes the inductive step.

(ii) $\Longrightarrow$ (i). Let $\mathfrak{m}$ be a maximal ideal of R. Now $\operatorname{Hom}_R(C, E(R/\mathfrak{m}))$ is C-injective and hence by assumption $\pi_i\big(\mathfrak{m}, \operatorname{Hom}_R(C, E(R/\mathfrak{m}))\big) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{m})$. Now by the same argument as in the proof of Theorem 4.3, we have $\operatorname{Ext}^i_{R_\mathfrak{m}}(k(\mathfrak{m}), C_\mathfrak{m}) = 0$ for all $i \neq \operatorname{ht}(\mathfrak{m})$, whence $C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$. □

Corollary 4.10. *The following statements hold true:*

(i) *If C is pointwise dualizing, then $C\text{-}\operatorname{id}_R(F(M)) \leq C\text{-}\operatorname{id}_R(M)$ for any cotorsion R-module M.*

(ii) *If $C\text{-}\operatorname{id}_R(F(M)) \leq C\text{-}\operatorname{id}_R(M)$ for any R-module M, then C is pointwise dualizing.*

Proof. (i). Assume that M is a cotorsion R-module. If $C\text{-}\operatorname{id}_R(M) = \infty$, then we are done. So assume that $C\text{-}\operatorname{id}_R(M) = n < \infty$. Then by Theorem 4.9, we have $F(M) = \prod T_\mathfrak{p}$ where $0 \leq \operatorname{ht}(\mathfrak{p}) \leq n$. Now the result follows by Lemma 4.1.

(ii). Assume that $\mathfrak{m}$ is a maximal ideal of R. We have to show that $C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$. Assume that $\mathbf{x}$ is a maximal R-sequence in $\mathfrak{m}$. Then $\operatorname{fd}_R(R/\mathbf{x}R) < \infty$, and $\operatorname{Ass}_R(C/\mathbf{x}C) = \{\mathfrak{m}\}$ since $\mathbf{x}$ is also a maximal C-sequence. Hence we have the equalities

$$\begin{aligned} C\text{-}\operatorname{fd}(C/\mathbf{x}C) &= \operatorname{fd}_R(\operatorname{Hom}_R(C, C/\mathbf{x}C)) \\ &= \operatorname{fd}_R(\operatorname{Hom}_R(C, C \otimes_R R/\mathbf{x}R)) \\ &= \operatorname{fd}_R(R/\mathbf{x}R) \\ &< \infty, \end{aligned}$$

in which the first equality is from Theorem 2.9(ii), and the third one holds because $R/\mathbf{x}R \in \mathcal{A}_C(R)$. Assume that E is an injective cogenerator. Set $(-)^\vee = \operatorname{Hom}_R(-, E)$. Then $C\text{-}\operatorname{id}_R((C/\mathbf{x}C)^\vee) < \infty$ by Lemma 2.10(ii). Now if F is the flat cover of $(C/\mathbf{x}C)^\vee$, then by assumption, we have $C\text{-}\operatorname{id}_R(F) < \infty$. Therefore, we have $C\text{-}\operatorname{fd}_R(F^\vee) < \infty$ by Lemma 2.10(i). Next, note that we have

$$C/\mathbf{x}C \hookrightarrow (C/\mathbf{x}C)^{\vee\vee} \hookrightarrow F^\vee.$$

Hence, the injective envelope of $C/\mathbf{x}C$ is a direct summand of $F^\vee$. Thus, in fact, $E(R/\mathfrak{m})$

is a direct summand of $F^\vee$, since $R/\mathfrak{m} \hookrightarrow C/\mathbf{x}C$. It follows that C-fd $_R(E(R/\mathfrak{m})) < \infty$, and hence we are done by Lemma 4.1, since $\mathfrak{m}$ was arbitrary. $\square$

Acknowledgments. We thank the referee for very careful reading of the manuscript and also for his/her useful suggestions.

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